Bending Analysis of Laminated Composite Plates Using Higher Order Theory Of 18 Degree Of Freedom Adopting Finite Element Approach

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Abstract— Laminated composite plates(LCP) are extensively used to solve special problems in engineering applications so bending, dynamic and stability behaviors are important to the designers. The paper aims at bending analysis of these plates with higher order theory. The application of higher-order theory that accounts for the realistic variation of in-plane and transverse displacements through the thickness for the static response analysis of thick multi-layered composite plates shall be studied. Code is developed using MATLAB with finite element formulation for 18 degrees of freedom with good agreement.

Index Terms— Laminated composite plates, bending analysis, transverse displacement, higher order theory, finite element, degrees of freedom, MATLAB.

1 Introduction

Composite plates are made by joining same or different material plates together in layers and laminated to fulfill the required properties. Laminated composite plates have unique properties than when compared to its constituent materials such as high stiffness to weight ratio, high strength to weight ratio, low maintenance, high corrosion resistance, durable, low specific weight, high specific strength and stiffness properties. Deformation of laminated plates is defined by coupling between bending and shear deformation.

2 Method

2.1 FEM

Finite element method is a mathematical method used to determine boundary value problems. In FEM the major element is divided into smaller ones called finite elements followed by solving them and then combining those to one major original initial problem. The division of an element is made by creating a mesh by joining certain number of nodes with each other in which that mesh constitutes the whole element. This project involves finite elements of mesh created by using nine nodes.

2.2 Displacement model

Based on the assumptions of displacement model, a higher order shear deformation theory (HSDT) is developed to analyze the stresses. The displacement model with EIGHTEEN degrees of freedom is in the following form:

The displacement model for unsymmetrical laminates,

$$u(x,y,z)=u_o(x,y)+z\Theta_x(x,y)+z^2u_o^*(x,y)+z^3\Theta_x^*(x,y)+z^4u_o^{**}(x,y)+z^5\Theta_x^{**}(x,y)$$

$$v(x,y,z)=v_o(x,y)+z\Theta_y(x,y)+z^2v_o^*(x,y)+z^3\Theta_y^*(x,y)+z^4v_o^{**}(x,y)+z^5\Theta_y^{**}(x,y)$$

$$w(x,y,z)=w_o(x,y)+z\Theta_z(x,y)+z^2w_o^*(x,y)+z^3\Theta_z^*(x,y)+z^4w_o^{**}(x,y)$$

Stress-Strain Relationship

$$\begin{Bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \gamma_{12}\\ \gamma_{23}\\ \gamma_{13}\end{Bmatrix}=\begin{bmatrix}1/E_1 & -\gamma_{21}/E_2 & \gamma_{31}/E_2 & 0 & 0 & 0\\ 1/E_{22} & -\gamma_{32}/E_3 & -\gamma_{12}/E_2 & 0 & 0 & 0\\ 1/E_3 & -\gamma_{13}/E_1 & -\gamma_{23}/E_2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1/G_{12} & 0 & 0\\ 0 & 0 & 0 & 0 & 1/G_{23} & 0\\ 0 & 0 & 0 & 0 & 0 & 1/G_{13}\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{12}\\ \tau_{23}\\ \tau_{13}\end{Bmatrix}$$

The strain corresponding to displacement model can be written as,

$$\epsilon_x = \epsilon_{xo} + zk_x + z^2\epsilon_{xo}^* + z^3k_x^* + z^4\epsilon_{xo}^{**} + z^5k_x^{**}$$

$$\epsilon_y = \epsilon_{yo} + zk_y + z^2\epsilon_{yo}^* + z^3k_y^* + z^4\epsilon_{yo}^{**} + z^5k_y^{**}$$

$$\epsilon_z = \epsilon_{zo} + zk_z^* + z^2\epsilon_{zo}^* + z^3k_z^{**} + z^4\epsilon_{zo}^{**}$$

$$\gamma_{xy} = \epsilon_{xyo} + zk_{xy} + z^2\epsilon_{xyo}^* + z^3k_{xy}^* + z^4\epsilon_{xyo}^{**} + z^5k_{xy}^{**}$$

$$\gamma_{yz} = \phi_y + zk_{yz} + z^2\phi_y^* + z^3k_{yz}^* + z^4\phi_y^{**} + z^5k_{yz}^{**}$$

$$\gamma_{xz} = \phi_x + zk_{xz} + z^2\phi_x^* + z^3k_{xz}^* + z^4\phi_x^{**} + z^5k_{xz}^{**}$$

Shape function for nine node –element

ISSN 2229-5518

1. $N_1 = \frac{1}{4}(\xi^2 - \xi)(\eta^2 - \eta)$

2. $N_2 = \frac{1}{4}(\xi^2 + \xi)(\eta^2 - \eta)$

3. $N_3 = \frac{1}{4}(\xi^2 + \xi)(\eta^2 + \eta)$

4. $N_4 = \frac{1}{4}(\xi^2 - \xi)(\eta^2 + \eta)$

5. $N_5 = \frac{1}{4}(1 - \xi^2)(\eta^2 - \eta)$

6. $N_6 = \frac{1}{4}(\xi^2 + \xi)(1 - \eta^2)$

7. $N_7 = \frac{1}{4}(1 - \xi^2)(\eta^2 - \eta)$

8. $N_8 = \frac{1}{4}(\xi^2 - \xi)(1 - \eta^2)$

9. $N_9 = \frac{1}{4}(1 - \xi^2)(\eta^2 - \eta)$

The components of stress resultant vector $\bar{\sigma}$ for the laminate with NL number of layers are defined as,

$$\begin{bmatrix} N_x & N_x^* & N_x^{**} & M_x & M_x^* & M_x^{**} \\ N_y & N_y^* & N_y^{**} & M_y & M_y^* & M_y^{**} \\ N_z & N_z^* & N_z^{**} & M_z^* & N_z^{**} & 0 \\ N_{xy} & N_{xy}^* & N_{xy}^{**} & M_{xy} & M_{xy}^* & M_{xy}^{**} \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{Bmatrix} (1 \ z^2 z^4 z z^3 z^5) dz$$

$$= \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{12} & Q_{22} & Q_{23} & Q_{24} \\ Q_{13} & Q_{23} & Q_{33} & Q_{34} \\ Q_{14} & Q_{24} & Q_{34} & Q_{44} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \Upsilon_{xy} \end{Bmatrix} (1 \ z^2 z^4 \ z \ z^3 z^5) dz$$

$$\begin{bmatrix} Q_x & Q_x^* & Q_x^{**} & S_x & S_x^* & S_x^{**} \\ Q_y & Q_y^* & Q_y^{**} & S_y & S_y^* & S_y^{**} \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} (1 \ z^2 z^4 \ z \ z^3 z^5) dz$$

$$\sigma_x = Q_{11}\epsilon_x + Q_{12}\epsilon_y + Q_{13}\epsilon_z + Q_{14}\Upsilon_{xy}$$

$$\sigma_y = Q_{12}\epsilon_x + Q_{22}\epsilon_y + Q_{23}\epsilon_z + Q_{24}\Upsilon_{xy}$$

$$\sigma_z = Q_{13}\epsilon_x + Q_{32}\epsilon_y + Q_{33}\epsilon_z + Q_{34}\Upsilon_{xy}$$

$$\tau_{xy} = Q_{14}\epsilon_x + Q_{42}\epsilon_y + Q_{43}\epsilon_z + Q_{44}\Upsilon_{xy}$$

$$\Upsilon_{xy} = \epsilon_{xyo} + zk_{xy} + z^2\epsilon_{xyo}^* + z^3k_{xy}^* + z^4\epsilon_{xyo}^{**} + z^5k_{xy}^{**}$$

$$\Upsilon_{yz} = \phi_y + zk_{yz} + z^2\phi_y^* + z^3k_{yz}^* + z^4\phi_y^{**} + z^5k_{yz}^{**}$$

$$\Upsilon_{xz} = \phi_x + zk_{xz} + z^2\phi_x^* + z^3k_{xz}^* + z^4\phi_x^{**} + z^5k_{xz}^{**}$$

$$H_i = \frac{1}{i}\left(z_{L+1}^i - z_L^i\right)$$

2.3 Load Condition

The condition considered for this project is Sinusoidal load with SS2 boundary condition where,

along x-axis, at y=0 and y=b

$v_o = w_o = v_o^* = v_o^{**} = w_o^* = w_o^{**} = \frac{\partial wo}{\partial x} = 0$

along y-axis, at x=0 and x=a

$u_o = w_o = u_o^* = u_o^{**} = w_o^* = w_o^{**} = \frac{\partial wo}{\partial y} = 0$

The deflection, internal stress resultants and stresses which are non-dimensional are obtained by multiplying the below mentioned constants,

$$m_1 = \frac{10E_2 h^3}{qa^4}; \ m_2 = \frac{1}{qa^2}; \ m_3 = \frac{1}{qa}$$

3 RESULT

The results are obtained for different problem conditions for different number of layers using MATLAB for different parameters. Load used was sinusoidal SS2 condition for square cross-ply with 2 layers (0°/90°), 3 layers (0°/90°/0°), 4 layers (0°/90°/90°/0°) and angle-ply with 4 layers (0°/45°/-45°/90°) which is simply supported.

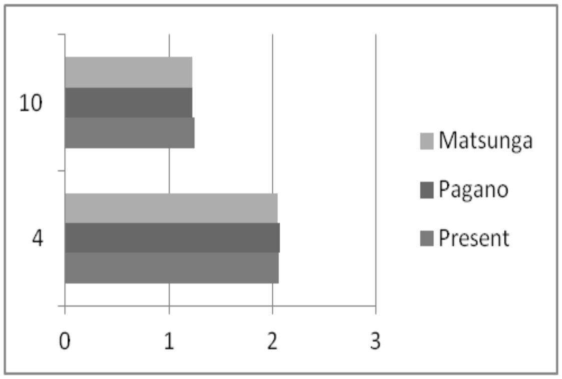
The results from published articles were compared. The scholarly articles of Pagano and Matsunga were compared with present results with same condition as mentioned from respective papers are mentioned in Table 1 for displacement (W). The results for problems with different a/h and E are mentioned in Table 2 and 3 for displacement (W), stresses (σ_x, σ_y, τ_{xy}). The numerical problems and results are as follows:

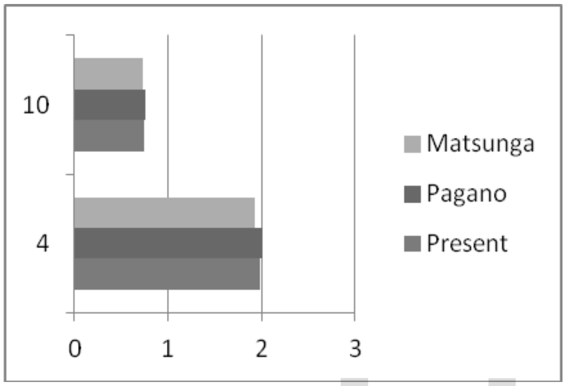
3.1 Problem 1: $E_1/E_2 = 25$; $G_{12} = G_{13} = 0.5$; $E_2 = E_3 = 1$; $\mu_{12} = \mu_{13} = 0.25$; cross-ply; SS2 condition

Data validation for W (displacement)

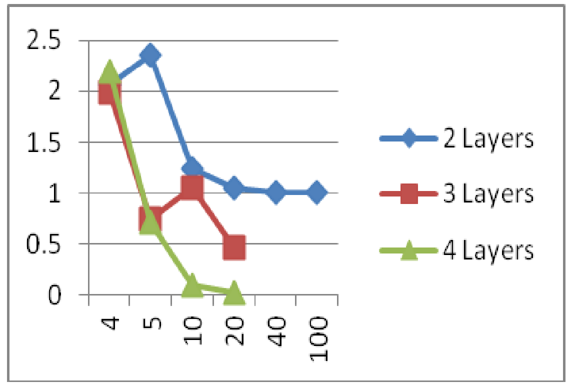
Table. No. 01 – Comparison with other papers (W)

Layers	a/h	Present	Pagano [1]	Matsunga [1]
2 (0°/90°)	4	2.0601	2.068	2.0483
	10	1.2438	1.2275	1.2243
3 (0°/90°/0°)	4	1.98541	2.0059	1.9228
	10	0.7503	0.753	0.7313

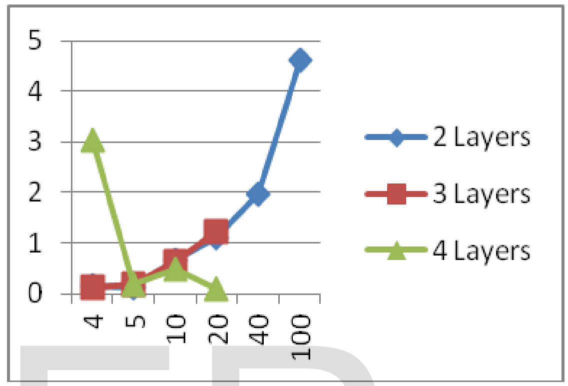




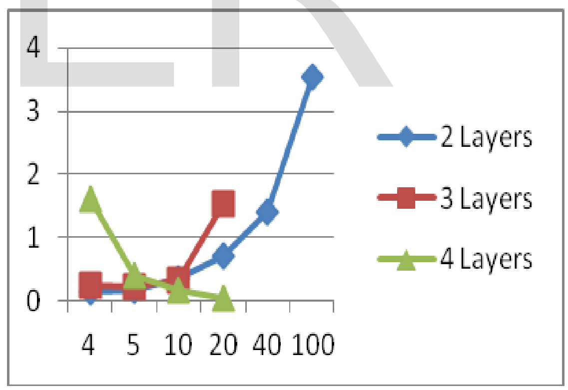
Comparison of displacement (W) with other papers for 2 Layers & 3 Layers respectively



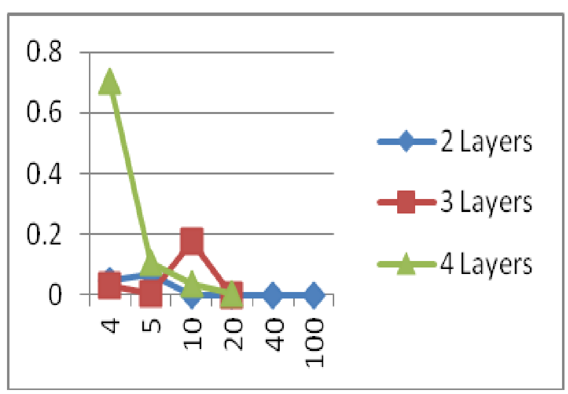
Displacement (W) vs a/h ratio for Problem 1



σ_x vs a/h ratio for Problem 1



σ_y vs a/h ratio for Problem 1



τ_{xy} vs a/h ratio Problem 1

3.2 Problem 1: $E_1/E_2 = 25$; $G_{12} = G_{13} = 0.5$; $E_2 = E_3 = 1$; $\mu_{12} = \mu_{13} = 0.25$; cross-ply; SS2 condition

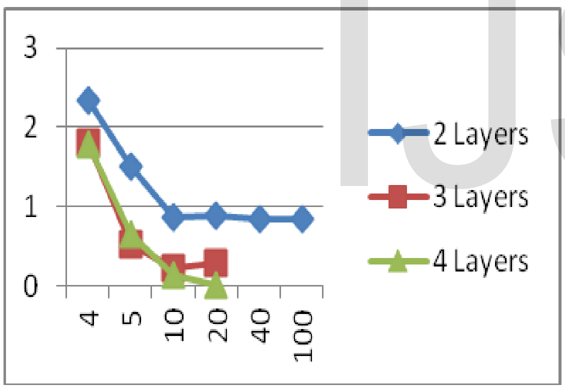
Table. No. 02 – Results for Problem 1

Layers	a/h	W x m_1	σ_x x m_2	σ_y x m_3	τ_{xy}
2 (0°/90°)	4	2.0601	0.1414	0.1328	0.04967
	5	2.3591	0.1237	0.16871	0.07071
	10	1.2438	0.6473	0.3476	0.00049
	20	1.05944	1.0972	0.702	0.000012
	40	1.0132	1.9768	1.4118	0.000048
	100	1.0078	4.6228	3.5449	0.00073
3 (0°/90°/0°)	4	1.98541	0.1206	0.2561	0.0293
	10	0.7503	0.1803	0.2263	0.00351
	20	1.05894	0.6174	0.3154	0.1773
	100	0.4642	1.2214	1.5227	0.000186
4 (0°/90°/90°/0°)	4	2.1933	3.0532	1.6076	0.7038
	10	0.7209	0.17136	0.4088	0.1079
	20	0.10017	0.4907	0.1739	0.03603
	40	0.03551	0.09086	0.0487	0.00614

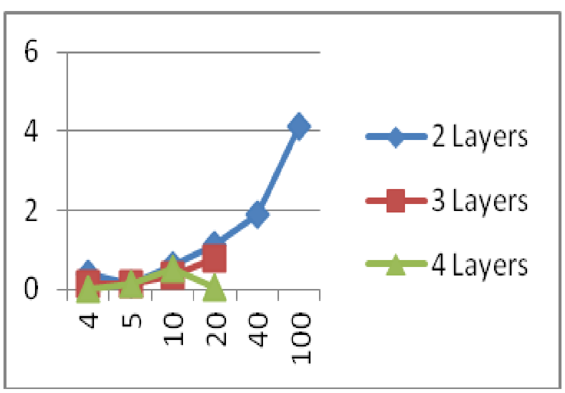
3.3 Problem 2: $E_1/E_2 = 40$; $G_{12} = G_{13} = 0.6$; $E_2 = E_3 = 1$; $\mu_{12} = \mu_{13} = 0.25$; cross-ply; SS2 condition

Table. No. 03 – Results for Problem 2

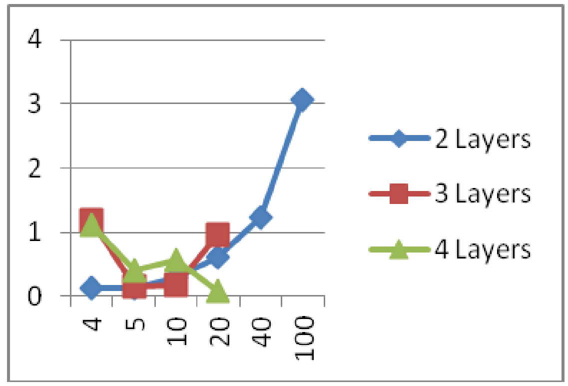
Layers	a/h	W x m_1	σ_x x m_2	σ_y x m_3	τ_{xy}
2 (0°/90°)	4	2.3307	0.3814	0.1232	0.0882
	5	1.5145	0.1446	0.1263	0.01476
	10	0.8692	0.5974	0.2823	0.0033
	20	0.8949	1.1083	6125	0.0014
	40	0.8505	1.8908	1.2371	0.000053
	100	0.8464	4.1185	3.0619	0.00131
3 (0°/90°/0°)	4	1.7971	0.1425	1.1808	0.073
	10	0.51716	0.1263	0.1609	0.000601
	20	0.2361	0.3488	0.1725	0.0433
	100	0.2985	0.7773	0.9658	0.0002
4 (0°/90°/90°/0°)	4	1.7935	0.0234	1.1261	0.01783
	10	0.6471	0.1548	0.4013	0.0039
	20	0.1323	0.5304	0.5598	0.2678
	40	0.01059	0.05212	0.09177	0.0238



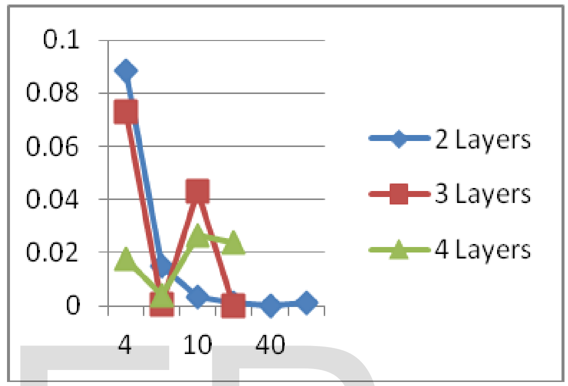
Displacement (W) vs a/h ratio Problem 2



σ_x vs a/h ratio for Problem 2



σ_y vs a/h ratio for Problem 2



τ_{xy} vs a/h ratio for Problem 2

3.3 Problem 3: $E_1/E_2 = 40$; $G_{12} = G_{13} = 0.6$; $E_2 = E_3 = 1$; $\mu_{12} = \mu_{13} = 0.25$; angle-ply; SS2 condition

Table. No. 04 – Results for Problem 3

Layers	a/h	W x m_1	σ_x x m_2	σ_y x m_3	τ_{xy}
4 (0°/45°/-45°/90°)	10	1.1551	0.5266	0.51917	0.2919
	50	0.8353	1.9284	1.91511	0.08399
	100	0.8304	3.594	3.5801	0.0792

4 Conclusion

The results show that the values are close to exact result values mentioned in articles of Pagano and Matsunga for problem 1 for displacement (W). Outcomes prove that the finite element method of higher order shear deformation theory can be effectively applied for laminated composite plates. The graph plotted for problem 1 and problem 2 outcomes respectively shows that the displacement (W) & τ_{xy} in both cases keeps on noticeably decreasing with increase in a/h ratio. The graph plotted for problem 1 and problem 2 outcomes shows that the stresses σ_x and σ_y in both cases keeps on increasing as a/h ratio increases. The results of problem 3 shows that σ_x and σ_y are almost similarly increasing while τ_{xy} and displacement (W) keeps on decreasing with increase in a/h ratio. On inte-

ISSN 2229-5518

grating equilibrium equations of elasticity over lamina thickness, the transverse stresses can be obtained.

REFERENCES

[1]. Assessment of a global higher-order deformation theory for laminated composite and sandwich plates- Hiroyuki Matsunaga, 2002.

[2]. N.J.Pagano, " Exact solutions for rectangular bidirectional composites and sandwich plates", J.Compos.Mat.,4,(1970) 20-34.

[3]. T Kant & B.S.Manjunatha, " An unsymmetric FRC laminate Co finite element model with 12 degrees of freedom per node", Eng. Comput., Vol 5, December 1988.

[4]. T. Kant and K. Swaminathan, "Analytical solutions for the static analysis of laminated composite and sandwich plates based on a higher order refined theory", composite structures 56 (2002) 329–344.

[5]. Reddy JN. Mechanics of laminated composite plates and shells: theory and analysis. CRC press; 2004.

[6]. Reddy J, Wang C, Lee K. Relationships between bending solutions of classical and shear deformation beam theories. Int J Solids Struct 1997.

[7]. Zhen W, Wanji C. An assessment of several displacement-based theories for the vibration and stability analysis of laminated composite and sandwich beams. Compos Struct 2008.

[8]. Khdeir A, Reddy J. An exact solution for the bending of thin and thick cross-ply laminated beams. Compos Struct 1997.

[9]. Vinson JR, Sierakowski RL. The behavior of structures composed of composite materials. Solid Mech Appl 2008